$$ {}^{A}_{B}H = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

from frame b to frame A

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}H \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$

$$ {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T} = {}^{B}_{A}R $$

$$ {}^{A}_{B}H^{-1} \neq {}^{A}_{B}H^{T} $$

Now how can we get the inverse of the transformation matrix.

$$ \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = {}^{B}_{A}H \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} $$

$$ = {}^{A}_{B}H^{-1} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} $$

how do we get this

$${}^AP = {}^A_BR\,{}^BP + {}^AP_{BORG}$$

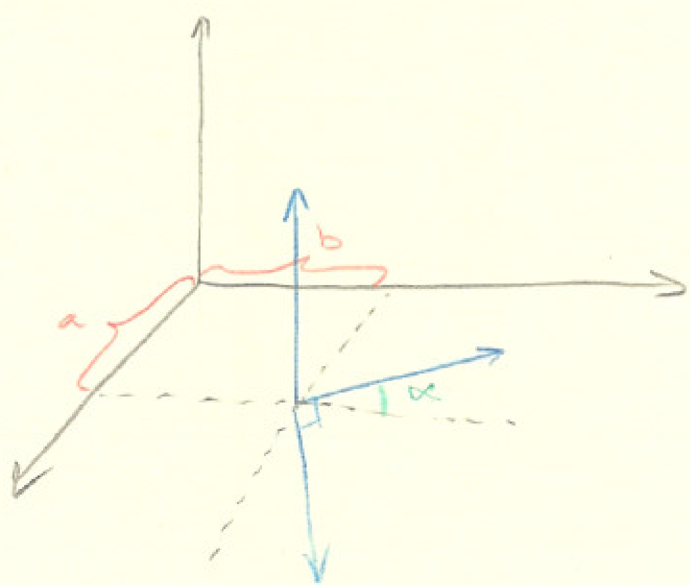
$${}^A_BR^T\,{}^AP = {}^A_BR^T\,{}^A_BR\,{}^BP + {}^A_BR^T\,{}^AP_{BORG}$$

$${}^BP = {}^A_BR^T\,{}^AP - {}^A_BR^T\,{}^AP_{BORG}$$

$$\begin{bmatrix} {}^BP \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} {}^A_BR^T & -{}^A_BR^T\,{}^AP_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]}\begin{bmatrix} {}^AP \\ 1 \end{bmatrix}$$

This is the inverse of the homogenous transformation matrix.

EX:



Here there is one translation and one rotation.

$$ {}^{A}_{B}H = \left[\begin{array}{c|c} {}^{A}_{B}R & \begin{matrix} a \\ b \\ 0 \end{matrix} \\ \hline 0\;0\;0 & 1 \end{array}\right] $$

$$ {}^{A}_{B}R = R_Z(\alpha) = \begin{bmatrix} C_\alpha & -S_\alpha & 0 \\ S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}_{B}H = \begin{bmatrix} C_\alpha & -S_\alpha & 0 & a \\ S_\alpha & C_\alpha & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}H = ? $$

$$ {}^{A}_{B}R^T = \begin{bmatrix} C_\alpha & S_\alpha & 0 \\ -S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ -{}^{A}_{B}R^T\,{}^{A}P_{BORG} = \begin{bmatrix} C_\alpha & S_\alpha & 0 \\ -S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} $$

$$= - \begin{bmatrix} a C_\alpha + b S_\alpha \\ -a S_\alpha + b S_\alpha \\ 0 \end{bmatrix}$$

$${}^B_A H = \begin{bmatrix} C_\alpha & S_\alpha & 0 & (-aC_\alpha - bS_\alpha) \\ -S_\alpha & C_\alpha & 0 & (+aS_\alpha - bC_\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EX: Consider 2 rotations, one about $\hat{z}$ by 30° and one about $\hat{x}$ by 30°

$$R_z(30°) = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_x(30°) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

$$R_z(30°) \cdot R_x(30°) = \begin{bmatrix} 0.87 & -.43 & 0.25 \\ 0.5 & 0.75 & -0.43 \\ 0 & 0.5 & 0.87 \end{bmatrix}$$

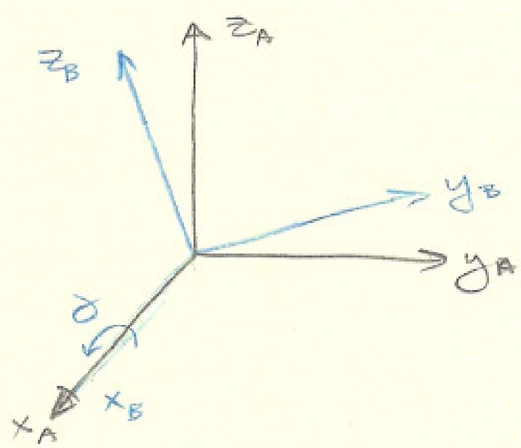
$$R_x(30°) \cdot R_z(30°) = \begin{bmatrix} 0.87 & -0.5 & 0 \\ 0.43 & 0.75 & -0.5 \\ 0.25 & 0.43 & 0.87 \end{bmatrix}$$

$$R_x(30°) R_z(30°) \neq R_z(30°) R_x(30°)$$

Types of Rotations

Start with frame B coinsitant $\overline{w}$ frame A

* Rotate B about $\hat{x}_A$ by angle γ
* Rotate B about $\hat{y}_A$ by angle β
* Rotate B about $\hat{z}_A$ by angle α



The we rotate about y_A and z_A.

This type of rotation is:

$\boxed{\text{X - Y - Z fixed angle}}$

$${}^{A}_{B}R_{xyz}(\gamma, \beta, \alpha) = R_x(\alpha) \cdot R_y(\beta) \cdot R_z(\gamma)$$

Note: The rotations are multiplied in the reverse order.

Note: For successive rotations about fixed frame, the composition law consists of multiplying the rotation matrices in the reverse order.

or xyz; does not matter.

Z - Y - X Euler Angles

Start w/ frame B coincident w/ frame A

* Rotate B about $\hat{z}_B$ by an angle α
* Rotate B about $\hat{y}_B$ by an angle β
* Rotate B about $\hat{x}_B$ by an angle γ

$${}^{A}_{B}R_{\hat{z}_B \hat{y}_B \hat{x}_B} = R_z(\alpha)\, R_y(\beta) \cdot R_x(\gamma)$$

Note: for 2 sucessive rotations with respect to the moving frame the composition law is multiplying the rotation matrices in direct order.

Z - Y - Z Euler Angles

Here we have 2 rotations about z and one about y.

Start w/ frame B coincident w/ frame A.

* Rotate B about $\hat{z}_B$ by α
* Rotate B about $\hat{y}_B$ by β
* Rotate B about $\hat{z}_B$ by γ

$$ {}^{A}_{B}R_{z_B y_B z_B}(\alpha, \beta, \gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma) $$